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Inverse variational problem for nonlinear equation in optics

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Abstract : Based on certain crucial observations on the symmetries of the Lagrangian and those of the corresponding equation of motion, an uncomplicated method is derived to deal with inverse problem of variational calculus. It is pointed out that the present approach will play a significant role in the study of pulse propagation through fibers. Some useful results are obtained for envelope equations in nonlinear couplers.

Keywords : Inverse problem, variational calculus, coupled pulses, optical fibers

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The solution of an inverse problem means discovering the cause of a known result. Studies in this area of investigation play a role in a wide variety of problems– ranging from physical sciences to diagnostics in medicine. In the following we cite some elementary examples.

(i) Gravitation :

Newton discovered the laws of gravitation from an observed set of data on the planetary motion.

(ii) Electrodynamics and optics :

The inverse problems of electrodynamics aim at determining the field sources and the shape of scattering bodies from indirect information about the electromagnetic field.

(iii) Quantum scattering theory :

Here, one tries to derive the potential or "interaction from a set of measured scattering data.

(iv) Seismology :

Seismology is the study of elastic vibration in earth crust using measurements obtained at the surface. Elastic fields carry information about the properties of the field sources and interior structure of the earth. Here, the inverse problem consists in finding the nature and location of the source which produces the seismic wave.

(v) Diagnostics in medicine or medical sciences :

When a doctor listens through the stethoscope, he perceives sound information and interprets it on the basis of his own experience. When an X-ray is taken, the way radiation is absorbed by certain parts of a human body is seen on a film and then interpreted by an expert. If the conventional X-ray photographs are taken for a large number of aspects, the information obtained is very large.

This presents difficulties in interpreting the information. In recent years, the interpretation is obtained by using computers. In particular, computer tomographs are used for medical diagnostics.

In this work, we shall deal with the inverse problem of variational calculus and try to answer the following questions.

(i) What is an inverse variational problem?

(ii) How is this problem solved? and

(iii) Why such solutions are important in nonlinear optics?

The calculus of variations deals with the problem of determining stationary values of functionals that characterize the time behaviour of mechanical systems goes by the name action. The process of minimizing the action functional for variation of the argument function is called the Hamilton's variational principle. Euler first discovered the necessary condition that a minimizing function must satisfy. It is well known that this condition gives rise to the so-called Euler-Lagrange equations and the function which satisfies the equations, referred to as Lagrangian function. In the calculus of variations, one is concerned with two types of problems, namely, the direct and inverse problem of mechanics. The direct problem is essentially the conventional one in which one first assigns a Lagrangian and then computes the equation of motion. As opposed to this, the inverse problem begins with the equations of motion and then construct a Lagrangian consistent with the variational principle. The inverse problem in classical dynamics was solved by Helmholtz [1] towards the end of the nineteenth century. There are two basic ingredients of the theory. One first makes a useful check on the existence of the Lagrangians for the system. This is done by examining the variational self-adjointness of the differential operator for the equation of mo-tion. The expression for the Lagrangian is then constructed using a homotopy formula [2].

It is not our intention to present details of the algebro-geometric theory derived by Helmholtz for the construction of Lagrangians. In contrast, we shall try to present an uncomplicated supplement for the method of Helmholtz. To introduce it, let us begin with a problem in Goldstein [3]. The problem has two parts.

(i) Show that if for a Lagrangian $L(q_i,\dot{q}_i,\ddot{q}_i,t)$ (often called the second order Lagrangian), Hamilton's principle holds with zero variation of both q_i and $\dot{q}_i$ at the end points, then the corresponding Euler-Lagrange's equations are

$$\frac{d^2}{dt^2}\left(\frac{\partial L}{\partial \ddot{q}_i}\right)-\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right)+\frac{\partial L}{\partial q_i}=0\,. \tag{1}$$

(ii) Apply this result to the Lagrangian

$$L=-\frac{m}{2}q\ddot{q}-\frac{k}{2}q^2 \tag{2}$$

and recognize the equation of motion.

The first part of problem was conjectured and solved by Euler [4]. For the second part we get

$$m\ddot{q}+kq=0\,. \tag{3}$$

Since, eq. (3) represents the equation of motion for a one-dimensional harmonic oscillator, one may reasonably ask : Can we obtain the well known harmonic oscillator Lagrangian from eq. (2)? It is easy to verify that the reduced Lagrangian

$$L_r=L+\frac{d}{dt}\left(\frac{m}{2}q\dot{q}\right) \tag{4}$$

corresponds to the well known result $L=\frac{m}{2}\dot{q}^2-\frac{1}{2}kq^2$. It is of interest to note that eq. (2) involves its own equation of motion because it can be written in the form

$$L=-\frac{1}{2}q(m\ddot{q}+kp). \tag{5}$$

Following the above example, we consider a general second-order equation of motion

$$\ddot{q}_i-f_i(q_i,\dot{q}_i)=0 \tag{6}$$

and venture to suggest that a second-order Lagrangian of eq. (6) is always given in the form

$$\bar{L}(q_i,\dot{q}_i,\ddot{q}_i)=\mu_i\left(\ddot{q}_i-f_i(q_i,\dot{q}_i)\right). \tag{7}$$

The second-order Lagrangian is related to the first-order or usual one by

$$L(q_i,\dot{q}_i)=\bar{L}(q_i,\dot{q}_i,\ddot{q}_i)+\frac{d}{dt}g(q_i,\dot{q}_i) \tag{8}$$

such that

$$\mu_i=-\frac{\partial g(q_i,\dot{q}_i)}{\partial \dot{q}_i}\,. \tag{9}$$

We have verified that eqs. (7), (8) and (9) are useful only when eq. (6) refers to a linear differential equation and these do not work even for a duffing oscillator

$$\ddot{x}+\omega^2x+4gax^3=0\,. \tag{10}$$

For example,

$$L=x(\ddot{x}+\omega^2x+4\alpha x^3)-\frac{d}{dt}(x\dot{x}) \tag{11}$$

does not represent the correct Lagrangian for eq. (10). If we postulate that the linear and nonlinear terms in the equation of motion contribute to the second-order Lagrangian with unequal weights and write

$$L=x\left[a\left(\ddot{x}+\omega^2x\right)+4b\alpha x^3\right]-\frac{d}{dt}(ax\dot{x}), \tag{12}$$

then eq. (12) would represent an admissible Lagrangian provided we determine the weight factors a and b by taking recourse to the use of Euler-Lagrange equations. This gives the correct Lagrangian

$$L=\frac{1}{2}\dot{x}^2-\frac{1}{2}\omega^2x^2-\alpha x^4. \tag{13}$$

So far, we have restricted our discussion to particle dynamics. However, the treatment can also be extended to field theory. A field theory is a generalization of classical mechanics in which the field variable $\phi(x,t)$ plays the role of dynamical variables $q_i(t)$, $i = 1,\dots, n$. The discrete index i, $1 \le i \le n$, now become the continuous variable $x \in \mathbb{R}^d$ and $\sum_{i=1}^{N}$ is replaced by $\int_{\mathbb{R}^d}$. In making this analogy, we would expect that for a classical field theory, the action functional W should be given as integral over space and time such that

$$W = \int_{\mathbb{R}^{d+1}} . \tag{14}$$

We should now demonstrate that our method is applicable to construct Lagrangian densities for two important equations of nonlinear optics. These are given by [5]

$$iu_x + u_{2t} + 2|u|^2 u = 0 \tag{15}$$

and the coupled set

$$iu_x = C_1 u_{2t} + 2\left(\alpha |u|^2 + \gamma |\upsilon|^2\right)\upsilon \tag{16a}$$

and

$$iu_x = C_1 \upsilon_{2t} + 2\left(\beta |u|^2 + \gamma |\upsilon|^2\right)\upsilon. \tag{16b}$$

Eq. (15) is called the nonlinear Schrödinger equation and governs the nonlinear pulse propagation in optical fibers. Equations in (16a) and (16b) represent the coupled envelope equations in nonlinear optical couplers.

In eq. (15), u stands for a complex-valued function. Thus, we can introduce a complex conjugate equation to write

$$-iu_x^* + u_{2t}^* + 2|u|^2 u^* = 0 . \tag{17}$$

In implementing the variational principle, u and u^* can be treated as separate variables. This allows us to express the Lagrangian density in the form :

$$\begin{aligned} \mathcal{L} = {} & u^*\left[a\left(iu_x + u_{2t}\right) + 2bu^*u^2\right] \\ & + u\left[a\left(-iu_x^* + u_{2t}^*\right) + 2buu^{*2}\right] \\ & - \frac{d}{dt}\left(au^*u_t + auu_t^*\right), \end{aligned} \tag{18}$$

which *via* the Euler-Lagrange equations

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial u_t} - \frac{\partial \mathcal{L}}{\delta u} = 0 \tag{19a}$$

and

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial u_t^*} - \frac{\partial \mathcal{L}}{\delta u^*} = 0. \tag{19b}$$

yields $a = (-1/2)$ and $b = (-1/4)$. Thus, the Lagrangian density is given by

$$\mathcal{L} = \frac{i}{2}(uu_x^* - u^*u_x) - |u|^4 + |u_t|^4 . \tag{20}$$

A similar treatment also applies for the coupled equations in (16) and we have the Lagrangian

$$\begin{aligned} \mathcal{L} = {} & i(u_x u^* - u_x^* u + \upsilon_x \upsilon^* - \upsilon_x^* \upsilon) \\ & + 2\left(C_1 |u_t|^2 + C_2 |\upsilon_t|^2\right) \\ & - 2\alpha |u|^4 - 2\gamma |\upsilon|^4 - 4\beta |u|^2 |\upsilon|^2 . \end{aligned} \tag{21}$$

Thus far, we have tried to answer, what is an inverse variational problem and how this problem is solved by using an uncomplicated method. Now let us turn our attention to the third question, namely, why such solutions are important in nonlinear optics. In other words, we must now indicate how the constructed Lagrangians are used to solve physical problems.

The Lagrangian presented in, eq. (20) supplemented by the Ritz optimization procedure [6], is often used to study the pulse propagation in optical fiber [5]. Here, the first variational action function is made to vanish for a suitably chosen trial function. To introduce this function, we begin by assuming that initially the pulse has a Gaussian form given by

$$u(0,t) = A_0 \exp\left(- \frac{}{2a_0^2}\right. \tag{22}$$

Here, A_0's represent the maximum amplitudes of the pulses $u(0,t)$ and a_0 are the corresponding pulse widths. We further assume that the pulse envelope maintains the Gaussian shape during subsequent evolution for $x > 0$. We, therefore, write

$$u(x,t) = A(x)\exp\left| - \frac{}{2a^2(x)} + ib(x)t^2\right. \tag{23}$$

where $A(x)$ is the complex amplitude, $a(x)$, the pulse width and $b(x)$, the frequency chirp. From eqs. (22) and (23), it is clear that $A(0) = A_0$ and $a(0) = a_0$. Understandably, the amplitude $A(x)$,pulse width $a(x)$ and frequency chirp $b(x)$ will all vary with the distance of propagation. Substituting eqs. (23) in (20), we get

$$\mathcal{L}_G = i\left[\left(A^* \frac{\partial A}{\partial x} - A\frac{\partial A^*}{\partial x}\right) + 2i|A|^2 \frac{\partial b}{\partial x} t^2\right.$$

$$\times \exp\left(-\frac{t^2}{2a^2}\right) - 2\alpha|A|^4 \exp\left(-\frac{2t^2}{a^2}\right)$$

$$\left. + 2|A|^2\left(\frac{1}{a^4} + 4b^2\right)t^2 \exp\left(-\frac{t^2}{a^2}\right)\right]. \tag{24}$$

Here, the subscript G on L indicates that we have inserted the Gaussian ansatz for $u(x,t)$ into the Lagrangian density. The action principle for $\mathcal{L}_G$ can be written in the form

$$\delta\int < L > dx = 0 \tag{25}$$

with

$$< L >= \int_{-\infty}^{\infty} \mathcal{L}_G dt. \tag{26}$$

The expression for $<L>$ is given by

$$< L >= \sqrt{\pi}\ \left[ia\left(A^* \frac{\partial A}{\partial x} - A\frac{\partial A^*}{\partial x}\right) - a^3|A|^2 \frac{\partial b}{\partial x}\right.$$

$$\left. -\sqrt{2a\alpha}\,|A|^4 + |A|^2 (a^{-1} + 4a^3b^2)\right]. \tag{27}$$

The variational principle as given in eq. (25) gives rise to a set of ordinary dif-ferential equations for $A(x)$, $a(x)$, and $b(x)$. These equations determine the dynamics of pulse propagation.

We conclude by noting that studies in propagation of coupled pulses in optical fibers, have crossed the era of exploration and reached that of exploitation. Thus, one may reasonably ask : What purpose will the present work serve? Our answer to this question is fairly straight forward. We believe that there are distinct advantages to viewing problems of physics within the the framework of simple analytical/semi-analytical models since many physical effects are then readily expressed and evaluated. For example, we began the present work by solving the inverse variational problem for the nonlinear Schrödinger equation and then studied the problem of pulse propagation by using a Gaussian trial functions and Ritz optimization procedure.

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